

# $G$ -COMPLETE REDUCIBILITY IN NON-CONNECTED GROUPS

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**ABSTRACT.** In this paper we present an algorithm for determining whether a subgroup  $H$  of a non-connected reductive group  $G$  is  $G$ -completely reducible. The algorithm consists of a series of reductions; at each step, we perform operations involving connected groups, such as checking whether a certain subgroup of  $G^0$  is  $G^0$ -cr. This essentially reduces the problem of determining  $G$ -complete reducibility to the connected case.

## 1. INTRODUCTION

Let  $G$  be a connected reductive linear algebraic group over an algebraically closed field of characteristic 0 or characteristic  $p > 0$ . Following Serre [13], we say a subgroup  $H$  of  $G$  is  *$G$ -completely reducible* ( $G$ -cr) if whenever  $H$  is contained in a parabolic subgroup  $P$  of  $G$ , then  $H$  is contained in some Levi subgroup of  $P$ . The definition extends to non-connected reductive  $G$  as well: one replaces parabolic and Levi subgroups with so-called Richardson parabolic and Richardson Levi subgroups respectively (see [2], [12] and Section 2).

Even if one is interested mainly in connected reductive groups, one must sometimes consider non-connected groups. For instance, natural subgroups of a connected group, such as normalizers and centralizers, are often non-connected. The notion of  $G$ -complete reducibility is much better understood in the connected case, e.g., see [1], [9], and [10]. In this paper we present an algorithm for determining whether a subgroup  $H$  of a non-connected reductive group  $G$  is  $G$ -cr. The algorithm consists of a series of reductions; at each step, we perform operations involving connected groups, such as checking whether a certain subgroup of  $G^0$  is  $G^0$ -cr. This essentially reduces the problem of determining  $G$ -complete reducibility to the connected case.

An important special case of the general problem described above is the following. Let  $H$  be a subgroup of  $G$ . We say  $H$  *acts on  $G^0$  by outer automorphisms* if for each  $1 \neq h \in H$ , conjugation by  $h$  gives a non-inner automorphism of  $G^0$ . In this case, we may identify  $H$  with a subgroup of  $\text{Out}(G^0)$ . Now suppose also that  $G^0$  is simple; then  $H$  is cyclic except for possibly when  $G^0$  is of type  $D_4$ . It is convenient when studying conjugacy classes in  $G^0$  to determine the fixed point set of a non-inner automorphism. See e.g., [11, Lem. 2.9] when  $H$  is cyclic and semisimple (that is,  $H$  is of order coprime to  $p$ ); note that if  $H$  is generated by a semisimple element then  $H$  is  $G$ -cr by Theorem 2.5, as semisimple conjugacy classes are closed. On the other hand, if  $H$  is cyclic and unipotent (that is,  $H$  is a  $p$ -group) then  $H$  can be  $G$ -cr or non- $G$ -cr.

We prove the following result, which gives a criterion for  $G$ -complete reducibility of  $H$ . It is an ingredient in our algorithm. In case  $H$  is cyclic, this is a special case of a recent result due to Guralnick and Malle, cf. Theorem 3.1.

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**Theorem** (Corollary 4.5). *Suppose  $G^0$  is simple and  $H$  acts on  $G^0$  by outer automorphisms. Then  $H$  is  $G$ -completely reducible if and only if  $C_{G^0}(H)$  is reductive.*

Our work fits into a study begun in our earlier papers [2], [3]. It was shown in [2, Thm. 3.10] that if  $H$  is a  $G$ -cr subgroup of  $G$  and  $N$  is a normal subgroup of  $H$  then  $N$  is also  $G$ -cr. In [3] we considered a complementary question: if  $H$  is a subgroup of  $G$ ,  $N$  is a normal subgroup of  $H$  and  $N$  is  $G$ -cr then under what hypotheses is  $H$  also  $G$ -cr? We gave an example (due to Liebeck) with  $H$  of the form  $M \times N$ , where  $M$  and  $N$  are both  $G$ -cr but  $H$  is not [3, Ex. 5.3]. We also showed this kind of pathological behaviour does not happen when  $G$  is connected and  $p$  is good for  $G$  [3, Thm. 1.3]. Here we study the above question in the case when  $N$  is the normal subgroup  $H \cap G^0$  of  $H$  (see the algorithm in Theorem 5.3).

## 2. PRELIMINARIES

**2.1. Notation.** Throughout, we work over an algebraically closed field  $k$  of characteristic  $p \geq 0$ ; we let  $k^*$  denote the multiplicative group of  $k$ . Let  $H$  be a linear algebraic group. By a subgroup of  $H$  we mean a closed subgroup. We let  $Z(H)$  denote the centre of  $H$  and  $H^0$  the connected component of  $H$  that contains 1. For  $h \in H$ , we let  $\text{Int}_h$  denote the automorphism of  $H$  given by conjugation with  $h$ . Frequently, we abbreviate  $\text{Int}_h(g)$  by  $h \cdot g$ . If  $S$  is a subset of  $H$  and  $K$  is a subgroup of  $H$ , then  $C_K(S)$  denotes the centralizer of  $S$  in  $K$  and  $N_K(S)$  the normalizer of  $S$  in  $K$ . Likewise, if  $S$  is a group of algebraic automorphisms of  $H$ , then we denote the fixed point subgroup of  $S$  in  $H$  by  $C_H(S)$ . If  $H$  acts on a set  $X$ , then we also write  $C_H(x)$  for the stabilizer of a point  $x \in X$  in  $H$ .

For the set of cocharacters (one-parameter subgroups) of  $H$  we write  $Y(H)$ ; the elements of  $Y(H)$  are the homomorphisms from  $k^*$  to  $H$ .

The *unipotent radical* of  $H$  is denoted  $R_u(H)$ ; it is the maximal connected normal unipotent subgroup of  $H$ . The algebraic group  $H$  is called *reductive* if  $R_u(H) = \{1\}$ ; note that we do not insist that a reductive group is connected. In particular,  $H$  is reductive if it is simple as an algebraic group. Here,  $H$  is said to be *simple* if  $H$  is connected and all proper normal subgroups of  $H$  are finite. The algebraic group  $H$  is called *linearly reductive* if all rational representations of  $H$  are semisimple.

Throughout the paper  $G$  denotes a reductive algebraic group, possibly non-connected.

**Definition 2.1.** Let  $H \subseteq G$  be a subgroup. We say that  $H$  *acts on  $G^0$  by outer automorphisms* if for every  $1 \neq h \in H$ , the automorphism  $\text{Int}_h|_{G^0}$  of  $G^0$  is non-inner, i.e., is not given by conjugation with an element of  $G^0$ . This is equivalent to the condition that  $H$  maps bijectively onto its image under the natural map  $G \rightarrow \text{Aut}(G^0) \rightarrow \text{Out}(G^0)$ .

**2.2.  $G$ -Complete Reducibility.** In [2, §6], Serre's original notion of  $G$ -complete reducibility is extended to include the case when  $G$  is reductive but not necessarily connected (so that  $G^0$  is a connected reductive group). The crucial ingredient of this extension is the use of so-called *Richardson-parabolic subgroups* ( *$R$ -parabolic subgroups*) of  $G$ . We briefly recall the main definitions here; for more details on this formalism, see [2, §6].

For a cocharacter  $\lambda \in Y(G)$ , the  *$R$ -parabolic subgroup* corresponding to  $\lambda$  is defined by

$$P_\lambda := \{g \in G \mid \lim_{a \rightarrow 0} \lambda(a)g\lambda(a)^{-1} \text{ exists}\}.$$

Here, for a morphism of algebraic varieties  $\phi : k^* \rightarrow X$ , we say that  $\lim_{a \rightarrow 0} \phi(a)$  exists provided that  $\phi$  extends to a morphism  $\widehat{\phi} : k \rightarrow X$ ; in this case we set  $\lim_{a \rightarrow 0} \phi(a) = \widehat{\phi}(0)$ .

Then  $P_\lambda$  admits a Levi decomposition  $P_\lambda = R_u(P_\lambda) \rtimes L_\lambda$ , where

$$L_\lambda = \{g \in G \mid \lim_{a \rightarrow 0} \lambda(a)g\lambda(a)^{-1} = g\} = C_G(\lambda(k^*)).$$

We call  $L_\lambda$  an *R-Levi subgroup* of  $P_\lambda$ . For an R-parabolic subgroup  $P$  of  $G$ , the different R-Levi subgroups of  $P$  correspond in this way to different choices of  $\lambda \in Y(G)$  such that  $P = P_\lambda$ ; moreover, the R-Levi subgroups of  $P$  are all conjugate under the action of  $R_u(P)$ . An R-parabolic subgroup  $P$  is a parabolic subgroup in the sense that  $G/P$  is a complete variety; the converse is true when  $G$  is connected, but not in general ([12, Rem. 5.3]).

*Remark 2.2.* For a subgroup  $H$  of  $G$ , there is a natural inclusion  $Y(H) \subseteq Y(G)$ . If  $\lambda \in Y(H)$ , and  $H$  is reductive, we can therefore associate to  $\lambda$  an R-parabolic subgroup of  $H$  as well as an R-parabolic subgroup of  $G$ . To avoid confusion, we reserve the notation  $P_\lambda$  for R-parabolic subgroups of  $G$ , and distinguish the R-parabolic subgroups of  $H$  by writing  $P_\lambda(H)$  for  $\lambda \in Y(H)$ . The notation  $L_\lambda(H)$  has the obvious meaning. Note that  $P_\lambda(H) = P_\lambda \cap H$  and  $L_\lambda(H) = L_\lambda \cap H$  for  $\lambda \in Y(H)$ . In particular,  $P_\lambda^0 = P_\lambda(G^0)$  and  $L_\lambda^0 = L_\lambda(G^0)$ . If  $\lambda \in Y(H)$  then the R-Levi subgroups of  $P_\lambda(H)$  are the  $R_u(P_\lambda(H))$ -conjugates of  $L_\lambda(H)$ ; in particular, any R-Levi subgroup of  $P_\lambda(H)$  is of the form  $L \cap H$  for some R-Levi subgroup  $L$  of  $P_\lambda$ .

For later use, we record the following way to construct R-Levi subgroups.

**Lemma 2.3.** *Let  $P$  be an R-parabolic subgroup of  $G$ , and let  $M$  be a Levi subgroup of  $P^0$ . Then  $N_P(M)$  is an R-Levi subgroup of  $P$ .*

*Proof.* We may choose  $\lambda \in Y(G)$  such that  $P = P_\lambda$ ,  $P^0 = P_\lambda(G^0)$  and  $M = L_\lambda(G^0) = L_\lambda^0$ . We have the Levi decomposition  $P = R_u(P_\lambda) \rtimes L_\lambda = R_u(P_\lambda^0) \rtimes L_\lambda$ . Since  $L_\lambda \subseteq N_P(L_\lambda^0)$  and  $R_u(P_\lambda^0) \cap N_P(L_\lambda^0) = 1$  (as  $R_u(P_\lambda^0)$  acts simply transitively on the set of Levi subgroups of  $P_\lambda^0$ ), we conclude that  $N_P(M) = N_P(L_\lambda^0) = L_\lambda$ .  $\square$

**Definition 2.4.** Suppose  $H$  is a subgroup of  $G$ . We say  $H$  is  *$G$ -completely reducible* ( $G$ -cr for short) if whenever  $H$  is contained in an R-parabolic subgroup  $P$  of  $G$ , then there exists an R-Levi subgroup  $L$  of  $P$  with  $H \subseteq L$ .

Since all parabolic subgroups (respectively all Levi subgroups of parabolic subgroups) of a connected reductive group are R-parabolic subgroups (respectively R-Levi subgroups of R-parabolic subgroups), Definition 2.4 coincides with Serre's original definition for connected groups [14].

Let  $H$  be a subgroup of  $G$  and let  $G \hookrightarrow \mathrm{GL}_m$  be an embedding of algebraic groups. Let  $\mathbf{h} \in H^n$  be a tuple of generators of the associative subalgebra of  $\mathrm{Mat}_m$  spanned by  $H$  (such a tuple exists for  $n$  sufficiently large). Then  $\mathbf{h}$  is called a *generic tuple* of  $H$ , see [5, Def. 5.4]. We recall the following geometric criterion for  $G$ -complete reducibility [5, Thm. 5.8]; it provides a link between the theory of  $G$ -complete reducibility and the geometric invariant theory of reductive groups.

**Theorem 2.5.** *Let  $H$  be a subgroup of  $G$  and let  $\mathbf{h} \in H^n$  be a generic tuple for  $H$ . Then  $H$  is  $G$ -completely reducible if and only if the orbit  $G \cdot \mathbf{h}$  under simultaneous conjugation is closed in  $G^n$ . In particular, if  $H = \langle h \rangle$  is a cyclic subgroup of  $G$ , then  $H$  is  $G$ -completely reducible if and only if the conjugacy class  $G \cdot h \subseteq G$  is closed.*

The following result has been proved with methods from geometric invariant theory (see [5, Def. 5.17]):

**Theorem 2.6.** *Assume that the subgroup  $H$  of  $G$  is not  $G$ -completely reducible. Then there exists an  $R$ -parabolic subgroup  $P$  of  $G$  with the following two properties:*

- (i)  $H$  is not contained in any  $R$ -Levi subgroup of  $P$ ,
- (ii)  $N_G(H) \subseteq P$ .

The geometric construction of  $P$  in [5, §4] is roughly as follows: there is a class of so-called *optimal destabilizing cocharacters*  $\Omega \subseteq Y(G)$  such that if  $\lambda \in \Omega$  then  $P := P_\lambda$  has properties (i) and (ii) as in Theorem 2.6. We call such an  $R$ -parabolic subgroup  $P$  of  $G$  an *optimal destabilizing  $R$ -parabolic subgroup* for  $H$ .

**2.3. Criteria for  $G$ -complete reducibility.** In this subsection we study criteria for  $G$ -complete reducibility in terms of some smaller group.

We first recall the following results about normal subgroups (cf. [3, Thm. 3.4, Cor. 3.7(ii)]).

**Theorem 2.7.** *Suppose that  $N \subseteq H$  are subgroups of  $G$  and  $N$  is normal in  $H$ .*

- (i) *If  $N$  is also normal in  $G$ , then  $H$  is  $G$ -completely reducible if and only if  $H/N$  is  $G/N$ -completely reducible.*
- (ii) *If  $H/N$  is linearly reductive, then  $H$  is  $G$ -completely reducible if and only if  $N$  is  $G$ -completely reducible.*

A homomorphism  $\pi : G_1 \rightarrow G_2$  is called *non-degenerate* provided that  $\ker(\pi)^0$  is a torus. The next result is contained in [2, Lem. 2.12 and §6]:

**Lemma 2.8.** *Let  $\pi : G_1 \rightarrow G_2$  be a non-degenerate epimorphism of reductive groups. Let  $H \subseteq G_1$  be a subgroup. Then  $H$  is  $G_1$ -completely reducible if and only if  $\pi(H)$  is  $G_2$ -completely reducible.*

As an immediate consequence, we obtain the following result which allows us to focus on the part of  $G$  that is effectively acting on  $G^0$ . Note that  $C_G(G^0)^0 \subseteq Z(G^0)^0$  is a torus.

**Corollary 2.9.** *Let  $H \subseteq G$  be a subgroup. Let  $\pi : G \rightarrow G/C_G(G^0)$  be the natural projection. Then  $H$  is  $G$ -completely reducible if and only if  $\pi(H)$  is  $\pi(G)$ -completely reducible.*

The following lemma gives two necessary conditions for a subgroup of  $G$  to be  $G$ -completely reducible, both of which can be checked in the connected group  $G^0$ .

**Lemma 2.10.** *Let  $H$  be a  $G$ -completely reducible subgroup of  $G$ . Then the following hold:*

- (i)  $H \cap G^0$  is  $G^0$ -completely reducible;
- (ii)  $C_{G^0}(H)$  is  $G^0$ -completely reducible.

*Proof.* (i). This is the content of [2, Lem. 6.10 (ii)]. (ii). Since  $H$  is  $G$ -cr, so is its centralizer  $C_G(H)$ , by [2, Cor. 3.17 and §6]. Now the result follows from part (i).  $\square$

Under the assumption that assertion (i) or (ii) of Lemma 2.10 holds, the next two lemmas allow us to replace the ambient group  $G$  with a potentially smaller subgroup  $M$ .

**Lemma 2.11.** *Let  $H$  be a subgroup of  $G$  and suppose that  $H \cap G^0$  is  $G^0$ -completely reducible. Then  $M = HC_{G^0}(H \cap G^0)$  is reductive. Moreover,  $H$  is  $G$ -completely reducible if and only if it is  $M$ -completely reducible.*

*Proof.* First note that  $M$  is a well-defined subgroup of  $G$ , since  $H$  normalizes  $C_{G^0}(H \cap G^0)$ . As  $C_{G^0}(H \cap G^0)$  is a  $G^0$ -cr subgroup (Lemma 2.10(ii)), it is reductive, by [13, Property 4]. The same is true for  $H \cap G^0$  by assumption. Hence  $(H \cap G^0)C_{G^0}(H \cap G^0)$  is the product of two reductive groups and thus is reductive. As this group contains  $M^0$  as a normal subgroup, the group  $M$  is reductive as well.

Now first suppose that  $H$  is not  $G$ -cr. Let  $\lambda \in Y(G) = Y(G^0)$  be an optimal destabilizing cocharacter for  $H$  in  $G$ . Then  $P_\lambda(G^0)$  contains the subgroup  $H \cap G^0$ , which is  $G^0$ -cr by assumption. Hence after replacing  $\lambda$  with an  $R_u(P_\lambda)$ -conjugate, we may assume that  $\lambda$  centralizes  $H \cap G^0$ . This implies that  $\lambda \in Y(M)$ , and  $H \subseteq P_\lambda(M) \subseteq P_\lambda$ . Since  $H$  is not contained in an R-Levi subgroup of  $P_\lambda$  (cf. Theorem 2.6), it is not contained in an R-Levi subgroup of  $P_\lambda(M)$ . We conclude that  $H$  is not  $M$ -cr.

Conversely, suppose that  $H$  is  $G$ -cr. Let  $\lambda \in Y(M)$  and suppose that  $H \subseteq P_\lambda(M)$ . To show that  $H$  is  $M$ -cr we need to show that  $H$  is contained in an R-Levi subgroup of  $P_\lambda(M)$ . Since  $H$  is  $G$ -cr, we can find a cocharacter  $\mu \in Y(G)$ , such that  $H \subseteq L_\mu \subseteq P_\mu = P_\lambda$ . In particular,  $\mu$  centralizes  $H \cap G^0$ , so that  $\mu \in Y(M)$ . Hence  $H \subseteq L_\mu(M) \subseteq P_\mu(M) = P_\lambda(M)$ , as required.  $\square$

*Remark 2.12.* In the situation of Lemma 2.11, the subgroup  $N = H \cap G^0$  of  $H$  is normal in  $M$ . Thus, by Theorem 2.7(i), we deduce that  $H$  is  $G$ -completely reducible if and only if  $H/(H \cap G^0)$  is  $M/(H \cap G^0)$ -completely reducible.

**Lemma 2.13.** *Let  $H$  be a subgroup of  $G$  and suppose that  $C_{G^0}(H)$  is  $G^0$ -completely reducible. Then  $M = HC_{G^0}(C_{G^0}(H))$  is reductive. Moreover,  $H$  is  $G$ -completely reducible if and only if it is  $M$ -completely reducible.*

*Proof.* We proceed as in the proof of Lemma 2.11: Again,  $M$  is well-defined since  $H$  normalizes  $C_{G^0}(C_{G^0}(H))$ . Moreover,  $M^0 \subseteq (H \cap G^0)C_{G^0}(C_{G^0}(H)) = C_{G^0}(C_{G^0}(H)) \subseteq M$ , yielding  $M^0 = (C_{G^0}(C_{G^0}(H)))^0$ . Since  $C_{G^0}(H)$  is  $G^0$ -cr by assumption, as before we may conclude that its centralizer is reductive, so that  $M$  is reductive.

Suppose that  $H$  is not  $G$ -cr. Let  $\lambda \in Y(G) = Y(G^0)$  be an optimal destabilizing cocharacter for  $H$  in  $G$ . By Theorem 2.6(ii),  $P_\lambda$  contains  $C_{G^0}(H)$ , which is  $G^0$ -cr. Thus we may again assume that  $\lambda$  centralizes  $C_{G^0}(H)$ , so that  $\lambda \in Y(M)$ . As before, we conclude that  $H$  is not  $M$ -cr.

Conversely, suppose that  $H$  is  $G$ -cr. Let  $\lambda \in Y(M)$  such that  $H \subseteq P_\lambda(M)$ . As  $\lambda$  evaluates in  $M^0$ , it centralizes  $C_{G^0}(H)$ , so that  $C_{G^0}(H)$  is contained in  $P_\lambda$ . On the other hand, since  $H$  is  $G$ -cr, we may find  $\mu \in Y(G)$  such that  $P_\mu = P_\lambda$  and such that  $\mu \in Y(C_{G^0}(H))$ . But then  $P_\mu(C_{G^0}(H)) = C_{G^0}(H)$ , which forces  $\mu$  to centralize  $C_{G^0}(H)$  [2, Lem. 2.4]. So  $\mu \in Y(M)$ , and  $H \subseteq L_\mu(M) \subseteq P_\mu(M) = P_\lambda(M)$ . As before, this shows that  $H$  is  $M$ -cr.  $\square$

*Remark 2.14.* Let  $H$  be a subgroup of  $G$  and let  $\pi: G \rightarrow G'$  be an isogeny. Then  $\pi(C_{G^0}(H)^0) = C_{G'^0}(\pi(H))^0$  (see the proof of [4, Lem. 3.1]), so  $C_{G'^0}(\pi(H))^0$  is reductive if and only if  $C_{G^0}(H)^0$  is.

We may write the connected reductive group  $G^0$  in the form

$$(2.15) \quad G^0 = SG_1 \cdots G_n,$$

where  $S$  is the radical of  $G^0$  and  $G_1, \dots, G_n$  are the simple components of the derived group of  $G^0$ . Any subgroup  $H$  of  $G$  acts via conjugation on the derived subgroup of  $G^0$  and hence



permutes the simple components. We obtain an induced action of  $H$  on the set of indices  $\{1, \dots, n\}$ . For  $1 \leq i \leq n$ , we use the shorthand

$$\widehat{G}_i = S \prod_{j \neq i} G_j$$

for the product of all factors in  $G^0$  above with the exception of  $G_i$ .

Our next lemma allows us to replace  $G$  with a collection of reductive groups whose identity components are simple.

**Lemma 2.16.** *Let  $H$  be a subgroup of  $G$ . For  $1 \leq i \leq n$ , let  $H_i := N_H(G_i)$  and let  $\pi_i : H_i G^0 \rightarrow H_i G^0 / \widehat{G}_i$  be the natural projection. Let  $I \subseteq \{1, \dots, n\}$  be a subset meeting each  $H$ -orbit. Then  $H$  is  $G$ -completely reducible if and only if  $\pi_i(H_i)$  is  $\pi_i(H_i G^0)$ -completely reducible for each  $i \in I$ .*

*Proof.* First note that, by construction,  $H_i$  and  $G^0$  both normalize the group  $\widehat{G}_i$ . Hence the map  $\pi_i$  is well-defined. Since  $H_i G^0$  is reductive, so is its image under  $\pi_i$ .

To prove the forward implication, suppose the assertion fails for some  $i \in I$ . Up to reordering the indices, we may assume that  $\pi_1(H_1)$  is not  $\pi_1(H_1 G^0)$ -cr and that  $H$  acts transitively on the set  $\{1, \dots, r\}$  for some  $r \geq 1$ . Let  $Q$  be an optimal destabilizing R-parabolic subgroup of  $\pi_1(H_1 G^0)$  for  $\pi_1(H_1)$ . To obtain a contradiction, we show that  $\pi_1(H_1)$  is contained in an R-Levi subgroup of  $Q$ . Consider the group  $Q^0$ . This is a parabolic subgroup of  $\pi_1(H_1 G^0)^0 = \pi_1(G^0) = \pi_1(G_1)$ , hence it is of the form  $Q^0 = \pi_1(P_1)$ , where  $P_1$  is a parabolic subgroup of  $G_1$ .

Since  $P_1$  contains the centre of  $G_1$  and  $\pi_1(P_1) = Q^0$  is normalized by  $\pi_1(H_1) \subseteq Q$ , it follows that  $P_1$  is normalized by  $H_1$ . Indeed, let  $h \in H_1$ . Then  $h \cdot P_1 \subseteq G_1$ . On the other hand,  $\pi_1(h \cdot P_1) = \pi_1(h) \cdot \pi_1(P_1) = \pi_1(P_1)$ . Since  $\ker(\pi_1) = \widehat{G}_1$ , this implies that  $h \cdot P_1 \subseteq P_1 \widehat{G}_1$ . We conclude that  $h \cdot P_1 \subseteq P_1(G_1 \cap \widehat{G}_1) = P_1$ , where we have used that the last intersection is central in  $G_1$ .

For  $2 \leq j \leq r$ , let  $h_j \in H$  be an element satisfying  $h_j \cdot G_1 = G_j$ . Let  $P_j = h_j \cdot P_1$ , which is a parabolic subgroup of  $G_j$ . Since we have just verified that  $H_1$  normalizes  $P_1$ , the definition of  $P_j$  does not depend on the choice of  $h_j$  that transports  $G_1$  to  $G_j$ .

We now consider the parabolic subgroup  $P = SP_1 \cdots P_n$  of  $G^0$ , where we take  $P_j = G_j$  for  $j > r$ . By construction,  $P$  is normalized by  $H$ . Indeed, any  $h \in H$  fixes  $S$  under conjugation, and permutes the groups  $G_1, \dots, G_n$ . If  $h$  maps  $G_i$  to  $G_j$  with  $i, j \in \{1, \dots, r\}$ , then  $(hh_i) \cdot G_1 = G_j$ , and hence  $h \cdot P_i = (hh_i) \cdot P_1 = P_j$ . So  $h$  also permutes the groups  $P_1, \dots, P_r$ , and thus normalizes  $P$ .

The group  $N_G(P)$  is thus an R-parabolic subgroup of  $G$  containing  $H$  with  $N_G(P)^0 = P$  (see [2, Prop. 6.1]). Since  $H$  is  $G$ -cr, it is contained in an R-Levi subgroup  $L$  of  $N_G(P)$ , hence it normalizes the Levi subgroup  $L^0$  of  $P$ . We may write  $L^0 = SL_1 \cdots L_n$  for certain Levi subgroups  $L_j$  of  $P_j$ . Then  $H_1$  normalizes  $L_1$ , since  $L_1 = L^0 \cap G_1$ . This forces  $\pi_1(H_1)$  to normalize a Levi subgroup of  $Q^0 = \pi_1(P_1)$ . By Lemma 2.3,  $\pi_1(H_1)$  is contained in an R-Levi subgroup of  $Q$ , yielding a contradiction.

To prove the reverse implication, we again assume after reordering the indices that  $1 \in I$  and that  $H$  permutes the set  $\{1, \dots, r\}$  transitively for some  $r \geq 1$ . Assume that  $H$  is not  $G$ -cr, and that  $Q \subseteq G$  is an optimal destabilizing R-parabolic subgroup of  $G$  containing  $H$ . Again we want to deduce that  $H$  is contained in an R-Levi subgroup of  $Q$ , contradicting our assumption.

Write  $Q^0 = SP_1 \cdots P_n$ , where the  $P_i$  are parabolic subgroups of  $G_i$ . Since  $H$  normalizes  $Q^0$ ,  $H_1$  normalizes  $P_1 = Q^0 \cap G_1$ . This means that  $\pi_1(H_1)$  is contained in  $N_{\pi_1(H_1 G^0)}(\pi_1(P_1))$ , and the latter is an R-parabolic subgroup of  $\pi_1(H_1 G^0)$ . Since  $1 \in I$ , the subgroup  $\pi_1(H_1)$  is contained in an R-Levi subgroup, say  $M$ , of this normalizer. Thus  $\pi_1(H_1)$  normalizes the Levi subgroup  $M^0$  of  $\pi_1(P_1)$ , which is hence of the form  $\pi_1(L_1)$  for some Levi subgroup  $L_1$  of  $P_1$ .

Since  $L_1$  contains the centre of  $G_1$ , as in the proof of the forward implication (where we have proved that  $H_1$  normalizes  $P_1$ ), we may conclude that  $H_1$  normalizes  $L_1$ . Choosing again elements  $h_j \in H$  with  $h_j \cdot G_1 = G_j$  for  $2 \leq j \leq r$ , we obtain well-defined Levi subgroups  $L_j := h_j \cdot L_1$  of  $h_j \cdot P_1 = P_j$ , where the latter equality follows from  $P_j = Q^0 \cap G_j$ . Proceeding similarly for the other  $H$ -orbits on  $\{1, \dots, n\}$  (each of which contains an element of  $I$  by assumption), we construct an  $H$ -stable Levi subgroup  $L = SL_1 \cdots L_n$  of  $Q^0$ . As before, by Lemma 2.3,  $H$  is contained in an R-Levi subgroup of  $Q$ , which gives the desired contradiction. This finishes the proof.  $\square$

### 3. CYCLIC SUBGROUPS

According to Theorem 2.5, a cyclic subgroup of  $G$  is  $G$ -completely reducible if and only if the conjugacy class of a generator is closed. With this characterization we reformulate a recent result of Guralnick and Malle, see [8, Thm. 2.3]:

**Theorem 3.1.** *Let  $H$  be a cyclic subgroup of  $G$ . Then  $H$  is  $G$ -completely reducible if and only if  $C_{G^0}(H)$  is reductive.*

By Lemma 2.10(ii), we deduce the following result.

**Corollary 3.2.** *Let  $H$  be a cyclic subgroup of  $G$ . Then  $C_{G^0}(H)$  is  $G^0$ -completely reducible if and only if it is reductive.*

Combining some of our previous reductions, we obtain the following weaker version of Theorem 3.1. This is of independent interest, as our arguments allow us to avoid the case-by-case considerations that are needed for the proof of Theorem 3.1.

**Theorem 3.3.** *Let  $H \subseteq G$  be a cyclic subgroup of prime order. Then  $H$  is  $G$ -completely reducible if and only if  $C_{G^0}(H)$  is reductive.*

*Proof.* The forward implication is clear, by Lemma 2.10(ii). Conversely, assume that  $C_{G^0}(H)$  is reductive. Since linearly reductive subgroups are  $G$ -cr (see [2, Lem. 2.6]), we may assume that  $k$  has positive characteristic  $p$  that coincides with the order of  $H$ .

We first show that  $C_{G^0}(H)$  is  $G^0$ -cr. Suppose this fails, and let  $P \subseteq G^0$  be an optimal destabilizing parabolic subgroup for  $C_{G^0}(H)$  in  $G^0$ . Then  $H$  normalizes  $P$ , by Theorem 2.6(ii). Let  $U$  be the unipotent radical of  $P$ . Then  $Z(U)$  has positive dimension and is normalized by  $H$  and  $P$ . Up to passing to a characteristic subgroup (the subgroup of elements of order dividing  $p$ ), we may assume that  $Z(U)$  has exponent  $p$ . Thus  $Z(U)$  has the structure of an  $\mathbb{F}_p$ -vector space of infinite dimension with an  $\mathbb{F}_p$ -linear  $H$ -action. As  $H$  is generated by an element  $h \in H$  of order  $p$ , there must be infinitely many fixed points of  $H$  on  $Z(U)$ . Indeed, on any  $H$ -stable finite dimensional subspace  $W$  of  $Z(U)$  the automorphism induced by  $h$  may be brought into Jordan normal form with block sizes bounded by  $p$  (the Jordan normal form exists as  $h$  has only eigenvalue  $1 \in \mathbb{F}_p$ ). As each block contributes at least  $p-1$  fixed points,  $H$  has at least  $(p-1)[\dim W/p]$  fixed points on  $W$ , and we can make

$\dim_{\mathbb{F}_p} W$  arbitrarily large. Taking the identity component of the  $H$ -fixed points on  $Z(U)$  hence yields a non-trivial, connected, normal, unipotent subgroup of  $C_{G^0}(H)$ , contradicting the reductivity assumption. We thus conclude that  $C_{G^0}(H)$  is  $G^0$ -cr.

By Lemma 2.13, it therefore suffices to show that  $H$  is  $M$ -cr, where  $M = HC_{G^0}(C_{G^0}(H))$ . Let  $M_1, \dots, M_r$  be the simple components of  $M^0$ . By definition of  $M$ ,  $C_{M_i}(H) \subseteq M_i \cap Z(M^0)$  is finite for each  $i$ . This forces  $H_i = N_H(M_i) = 1$  for each  $i$ . Indeed, since  $H$  has prime order,  $H_i \neq 1$  would yield that  $H$  normalizes  $M_i$ , but due to a result of Steinberg (cf. [16, Thm. 10.13]), no non-trivial cyclic group can act on a simple group via algebraic automorphisms with only finitely many fixed points. By Lemma 2.16, we conclude that  $H$  is  $M$ -cr, as required.  $\square$

#### 4. OUTER AUTOMORPHISMS FOR $D_4$

In this section, let  $D_4$  denote an adjoint simple group of type  $D_4$ . Amongst the simple groups  $D_4$  has the largest outer automorphism group, in that  $\text{Out}(D_4) \cong S_3$ , the symmetric group on 3 letters. We may identify  $\text{Out}(D_4)$  with the set of graph automorphisms in  $\text{Aut}(D_4)$  induced by the symmetries of the Dynkin diagram. However, there are other subgroups isomorphic to  $S_3$  in  $\text{Aut}(D_4)$  that act via outer automorphisms. As this is the only situation where outer automorphisms of a simple group arise that is not covered by Theorem 3.3, we treat this case separately in this section.

Let  $T$  be a maximal torus of  $D_4$  with associated root system  $\Phi$ . Let  $\Delta = \{\alpha, \beta, \gamma, \delta\}$  be a set of simple roots for  $\Phi$ , where  $\delta$  is the unique simple root that is non-orthogonal to every other simple root. Let  $\lambda = \omega_\delta^\vee \in Y(T)$  be the fundamental dominant coweight determined by  $\langle \alpha, \lambda \rangle = \langle \beta, \lambda \rangle = \langle \gamma, \lambda \rangle = 0$ ,  $\langle \delta, \lambda \rangle = 1$ . For  $\epsilon \in \Phi$  we denote by  $u_\epsilon : \mathbb{G}_a \rightarrow U_\epsilon$  a fixed root homomorphism onto the corresponding root subgroup of  $G$ .

Let  $\sigma \in \text{Aut}(D_4)$  be the triality graph automorphism determined by requiring that for all  $c \in k$ ,

$$\sigma(u_\alpha(c)) = u_\beta(c), \quad \sigma(u_\beta(c)) = u_\gamma(c), \quad \sigma(u_\gamma(c)) = u_\alpha(c), \quad \sigma(u_\delta(c)) = u_\delta(c).$$

Then  $C_{D_4}(\sigma)$  is a simple group of type  $G_2$ . In fact,  $\tilde{T} = C_T(\sigma)$  is a maximal torus of  $C_{D_4}(\sigma)$ , and  $\tilde{\alpha} = \alpha|_{\tilde{T}} = \beta|_{\tilde{T}} = \gamma|_{\tilde{T}}$  and  $\tilde{\beta} = \delta|_{\tilde{T}}$  form a pair of simple roots with respect to  $\tilde{T}$ , with corresponding root groups given by  $u_{\tilde{\alpha}}(c) = u_\alpha(c)u_\beta(c)u_\gamma(c)$ ,  $u_{\tilde{\beta}}(c) = u_\delta(c)$ . Since  $\lambda$  evaluates in  $\tilde{T}$ , we may regard it as an element of  $Y(\tilde{T})$ ; we denote this element by  $\tilde{\lambda}$ . We have  $\langle \tilde{\alpha}, \tilde{\lambda} \rangle = 0$ ,  $\langle \tilde{\beta}, \tilde{\lambda} \rangle = 1$ .

We begin with a detailed description of triality in the particular case where the ground field has characteristic three, using the results of [6] and [7].

**Proposition 4.1.** *Assume that  $p = 3$ . In  $\text{Aut}(D_4)$  there are exactly two conjugacy classes of cyclic groups of order three generated by outer automorphisms. Let  $\langle \sigma_1 \rangle$ ,  $\langle \sigma_2 \rangle$  be representatives of the respective classes, and let  $M_i = C_{D_4}(\sigma_i)$  ( $i = 1, 2$ ). Then we may choose the labelling such that the following holds:*

- (i)  $M_1$  is a simple group of type  $G_2$ ; moreover  $\text{Aut}(D_4) \cdot \sigma_1$ , the orbit of  $\sigma_1$  under conjugation, is closed in  $\text{Aut}(D_4)$ .
- (ii)  $M_2$  is an 8-dimensional group with 5-dimensional unipotent radical and corresponding reductive quotient isomorphic to  $\text{SL}_2$ ; the orbit  $\text{Aut}(D_4) \cdot \sigma_2$  is not closed and contains  $\sigma_1$  in its closure.



- (iii) We may take  $\sigma_1 = \sigma$ .
- (iv) Let  $u = u_{\alpha+\beta+\gamma+2\delta}(1)$ . Then we may take  $\sigma_2 = \sigma u$ .
- (v) With the choices in (iii) and (iv), we have  $M_2 = C_{M_1}(u) = \langle U_{\tilde{\alpha}}, U_{-\tilde{\alpha}} \rangle \ltimes R_u(P_{\tilde{\lambda}}) \subseteq P_{\tilde{\lambda}}$ , where  $P_{\tilde{\lambda}}$  denotes  $P_{\lambda}(M_1)$ .

*Proof.* By [6, Cor. 6.5, Thm. 9.1], there are precisely two conjugacy classes of cyclic groups of order three generated by outer automorphisms, which are denoted by type I and type II, respectively. They are distinguished by the structure of the corresponding fixed point groups, where type I yields a group of type  $G_2$ , whereas type II in characteristic 3 gives a group with the structure described in (ii) (see [6, §9] together with [7, Thm. 7]). This implies the first statements of (i) and (ii), as well as (iii).

Working in the algebraic group  $\text{Aut}(D_4)$ , using  $\sigma(\lambda) = \lambda$  and  $\langle \alpha + \beta + \gamma + 2\delta, \lambda \rangle = 2 > 0$  we compute that

$$\lim_{a \rightarrow 0} \lambda(a) \sigma u \lambda(a)^{-1} = \sigma \lim_{a \rightarrow 0} \lambda(a) u \lambda(a)^{-1} = \sigma,$$

which proves that  $\sigma$  is contained in the closure of the orbit through  $\sigma u$ . By [5, Thm. 3.3], to show that  $\sigma$  and  $\sigma u$  are not conjugate it is enough to show that they are not  $D_4$ -conjugate. This non-conjugacy follows from [15, §I, Prop. 3.2], as  $\sigma$  and  $\sigma u$  are given in *loc. cit.* as class representatives for distinct unipotent classes in  $\sigma D_4$ . Moreover, by [15, §II, Lem. 1.15], the orbit through  $\sigma$  is closed. This proves (iv) and the remaining assertions of (i) and (ii).

To prove (v), we first note that  $R_u(P_{\tilde{\lambda}})$  consists of the root groups for the roots  $\tilde{\beta}, \tilde{\alpha} + \tilde{\beta}, 2\tilde{\alpha} + \tilde{\beta}, 3\tilde{\alpha} + \tilde{\beta}$  and  $3\tilde{\alpha} + 2\tilde{\beta}$ . In particular, the semi-direct product  $\langle U_{\tilde{\alpha}}, U_{-\tilde{\alpha}} \rangle \ltimes R_u(P_{\tilde{\lambda}})$  has dimension 8 and is contained in  $C_{M_1}(u) = C_{M_1}(u_{3\tilde{\alpha}+2\tilde{\beta}}(1))$ . As  $\tilde{\lambda}$  centralizes  $\pm\tilde{\alpha}$ , the semi-direct product is also contained in  $P_{\tilde{\lambda}}$ . Since clearly  $C_{M_1}(u) \subseteq M_2$ , the assertion (v) follows by comparing dimensions. This finishes the proof.  $\square$

We can now characterize  $G$ -complete reducibility in the case where  $G^0 = D_4$  and  $H$  maps isomorphically onto the full group of outer automorphisms of  $D_4$ . The following result is the analogue of Theorem 3.3.

**Theorem 4.2.** *Let  $H$  be a subgroup of  $G$ . Assume that  $G^0 = D_4$  and  $H \cong S_3$  acts by outer automorphisms on  $G^0$ . Then  $H$  is  $G$ -completely reducible if and only if  $C_{G^0}(H)$  is reductive.*

*Proof.* The forward implication is clear by Lemma 2.10. Conversely, assume that  $H$  is not  $G$ -cr. Let  $h \in H$  be an element of order 3, so that  $K = \langle h \rangle$  is a normal subgroup of index 2 in  $H$ . By the assumption on  $H$ , the map  $\pi : G \rightarrow \text{Aut}(D_4), g \mapsto \text{Int}(g)|_{G^0}$  is surjective. Since  $\ker(\pi) = C_G(G^0)$ ,  $\pi$  is an isogeny. Hence  $\pi(H)$  is not  $\text{Aut}(D_4)$ -cr, by Lemma 2.8. It now follows from Remark 2.14 that we can take  $G$  to be  $\text{Aut}(D_4)$ .

First assume that  $p = 3$ . Let  $M_1 = C_{D_4}(\sigma)$  and  $M_2 = C_{D_4}(\sigma u)$  with notation as in Proposition 4.1. Then  $K$  is a normal subgroup of order 3 and index 2 in  $H$ . Since  $p = 3$  is coprime to 2, we have by Theorem 2.7(ii) that  $K$  is not  $\text{Aut}(D_4)$ -cr. This implies (by Theorem 2.5) that the orbit  $\text{Aut}(D_4) \cdot h$  is not closed, whence by Proposition 4.1 there exists  $g \in G$  with  $ghg^{-1} = \sigma u$ . Replacing  $H$  with  $gHg^{-1}$ , we may assume that  $h = \sigma u$ . Let  $s \in H$  be an element of order 2 such that  $h$  and  $s$  generate  $H$ . Let  $\tau \in \text{Aut}(D_4)$  be the graph automorphism of order 2 determined by  $s$ , i.e., the graph automorphism that induces the same element as  $s$  in  $\text{Out}(D_4)$ . Let  $t = \beta^\vee(-1) \in \tilde{T} \subseteq M_1$ . Since  $\tau$  and  $\sigma$  fix  $M_1$ , both elements commute with  $t$  and  $u$ . Moreover, by construction  $tut = u^{-1}$ . This implies that  $\tau t$  has order 2 and  $(\tau t)(\sigma u)(\tau t) = \tau \sigma \tau u^{-1} = \sigma^{-1} u^{-1} = (\sigma u)^{-1}$ . As  $\tau t$  induces the same

element as  $s$  in  $\text{Out}(D_4)$ , we can find  $x \in D_4$  with  $s = \tau tx$ . We conclude that both pairs of elements  $h = \sigma u, s = \tau tx$  as well as  $\sigma u, \tau t$  generate a group isomorphic to  $S_3$ . In particular,  $x(\sigma u)x^{-1} = (\tau t)(\tau tx)(\sigma u)(\tau tx)^{-1}(\tau t) = (\tau t)(\sigma u)^{-1}(\tau t) = \sigma u$ . Thus  $x \in M_2 \subseteq M_1$  (cf. Proposition 4.1(v)), so that  $s = \tau tx$  normalizes  $M_1$ . As  $M_1$  is simple of type  $G_2$ , it has no outer automorphisms. Therefore we may find  $s' \in M_1$  with  $\text{Int}(s)|_{M_1} = \text{Int}(s')|_{M_1}$ . Since  $M_1$  is adjoint,  $s'$  is of order 2. Now

$$(4.3) \quad C_{G^0}(H) = C_{M_2}(s').$$

Since  $s = \tau tx$  normalizes  $M_2$ ,  $s$  normalizes  $N_{M_1}(M_2) = P_\lambda$  (see Proposition 4.1(v)). Hence  $s' \in P_\lambda$  and  $C_{M_2}(s') \subseteq P_\lambda$ . Up to conjugation in  $P_\lambda$  we may thus assume  $s' \in \tilde{T}$ . As  $\tilde{T}$  is generated by the images of  $\tilde{\alpha}^\vee$  and  $\tilde{\beta}^\vee$ , this reduces the possibilities to  $s' \in \{\tilde{\alpha}^\vee(-1), \tilde{\beta}^\vee(-1), \tilde{\alpha}^\vee(-1)\tilde{\beta}^\vee(-1)\}$ . But then  $s'$  centralizes  $U_{3\tilde{\alpha}+2\tilde{\beta}}$ , or  $U_{\tilde{\beta}}$ , or  $U_{\tilde{\alpha}+\tilde{\beta}}$  respectively. We deduce that  $C_{M_2}(s') \cap R_u(P_\lambda)$  is positive-dimensional in each case. Thus  $C_{M_2}(s')$  is not reductive, as required.

Now let  $p \neq 3$ . Then the subgroup  $K$  of  $H$  of order 3 is linearly reductive, in particular it is  $G$ -cr and  $C_{G^0}(K)$  is reductive. Moreover, the group  $C_{G^0}(K)$  is connected being the fixed point group under a triality automorphism (cf. [6, §9]). Let  $M = HC_{G^0}(K)$ . By [3, Thm. 3.1(b)(ii)] applied to  $K \subseteq H \subseteq M$ , we deduce that  $H$  is not  $M$ -cr. Since  $K$  is normal in  $M$ , by Theorem 2.7(i),  $H/K$  is not  $M/K$ -cr. But  $H/K$  is cyclic of order 2, so we may apply Theorem 3.3 to conclude that  $C_{(M/K)^0}(H/K)$  is not reductive. By construction,  $(M/K)^0 \cong C_{G^0}(K)$  and

$$(4.4) \quad C_{(M/K)^0}(H/K) \cong C_{G^0}(H).$$

This finishes the proof.  $\square$

Having settled the case of  $D_4$ , we can combine Theorems 3.3 and 4.2 to characterize  $G$ -complete reducibility in case  $G^0$  is simple and the subgroup  $H$  acts by outer automorphisms.

**Corollary 4.5.** *Let  $H \subseteq G$  be a subgroup acting on  $G^0$  by outer automorphisms. Assume that  $G^0$  is simple. Then  $H$  is  $G$ -completely reducible if and only if  $C_{G^0}(H)$  is reductive.*

*Proof.* We may assume that  $G^0$  is adjoint (cf. the first paragraph of the proof of Theorem 4.2). Since  $H$  acts via outer automorphisms, we may identify it as an abstract group with a subgroup of  $\text{Out}(G^0)$ , the finite group of outer automorphisms of  $G^0$ . As  $G^0$  is simple,  $\text{Out}(G^0)$  is either simple of prime order or  $G^0$  is of type  $D_4$  and  $\text{Out}(G^0) \cong S_3$ . The result now follows from Theorems 3.3 and 4.2.  $\square$

*Remark 4.6.* In the situation of Corollary 4.5, we always have  $\dim C_{G^0}(H) > 0$ . This follows again from the theorem of Steinberg ([16, Thm. 10.13]) in case  $H$  is cyclic. The general case follows from the identities for  $C_{G^0}(H)$  in (4.3) and (4.4).

## 5. THE ALGORITHM

We return to the general situation where  $H \subseteq G$  is a subgroup of a possibly non-connected reductive group. In this section, we are going to establish an algorithm that reduces the question of whether  $H$  is  $G$ -cr to the question of whether certain subgroups of certain *connected* reductive groups are  $G$ -completely reducible.

We start with a proposition that recasts some of our earlier results as operations for a potential algorithm. We say that the pair  $(H, G)$  is *completely reducible* provided that  $H$  is  $G$ -completely reducible.

**Proposition 5.1.** *Let  $(H, G)$  be a pair consisting of a reductive group  $G$  and a subgroup  $H \subseteq G$ . Then each of the following operations replaces  $(H, G)$  with pairs of the same form (i.e., consisting of a group and a reductive group containing it as a subgroup):*

- (O1) *Let  $\pi : G \rightarrow G/C_G(G^0)$  be the canonical projection. Replace  $(H, G)$  with  $(\pi(H), \pi(G))$ .*
- (O2) *Let  $G^0 = SG_1 \cdots G_n$  be the decomposition as in (2.15), and let  $H_i, \pi_i$  for  $1 \leq i \leq n$  be defined as in Lemma 2.16. Replace  $(H, G)$  with the pairs*

$$(\pi_1(H_1), \pi_1(H_1 G^0)), \dots, (\pi_n(H_n), \pi_n(H_n G^0)).$$

- (O3) *If  $H \cap G^0$  is  $G^0$ -completely reducible, replace  $(H, G)$  with*

$$(H/(H \cap G^0), HC_{G^0}(H \cap G^0)/(H \cap G^0)).$$

Moreover,  $H$  is  $G$ -completely reducible if and only if each of the pairs obtained through one of these operations is completely reducible.

*Proof.* The results follow from Corollary 2.9, Lemma 2.16 and Remark 2.12.  $\square$

**Remark 5.2.** In the situation of (O2), suppose we are given a set  $I$  as in Lemma 2.16. Then it is enough to replace  $(H, G)$  with the pairs  $(\pi_i(H_i), \pi_i(H_i G^0))_{i \in I}$  in (O2).

We are now in a position to give an algorithm that determines whether  $H$  is  $G$ -completely reducible.

**Theorem 5.3.** *Let  $H$  be a subgroup of  $G$ . The following algorithm, starting with the pair  $(H, G)$ , reduces the question of whether  $H$  is  $G$ -completely reducible in a finite number of steps to questions of complete reducibility in connected groups:*

**Algorithm.** *Input: a pair  $(H', G')$ , where  $H'$  is a subgroup of a reductive group  $G'$ .*

- Step 1 *If  $G'^0$  is not simple, or  $C_{G'}(G'^0) \neq 1$ , apply (O2) and then (O1) to each of the newly obtained pairs. Restart instances of the algorithm for each of the new pairs. Then  $H'$  is  $G'$ -cr if and only if each of these pairs turns out to be completely reducible.*
- Step 2 *Identify  $H'/(H' \cap G'^0)$  with a subgroup of  $\text{Out}(G'^0)$ . Let  $n \in \{1, 2, 3, 6\}$  be the order of  $H'/(H' \cap G'^0)$ . If  $p$  does not divide  $n$ ,  $H'$  is  $G'$ -cr if and only if  $H' \cap G'^0$  is  $G'^0$ -cr, and the algorithm stops.*
- Step 3 *If  $H' \cap G'^0$  is not  $G'^0$ -cr, the algorithm stops with the conclusion that  $H'$  is not  $G'$ -cr.*
- Step 4 *If  $H' \cap G'^0 = 1$ ,  $H'$  is  $G'$ -cr if and only if  $C_{G'^0}(H')$  is reductive. The algorithm stops.*
- Step 5 *If  $1 \neq H'/(H' \cap G'^0) \not\cong S_3$ , let  $M = H' C_{G'^0}(H' \cap G'^0)/(H' \cap G'^0)$ . Then  $H'$  is  $G'$ -cr if and only if  $C_{M^0}(H'/(H' \cap G'^0))$  is  $M^0$ -cr. The algorithm stops.*
- Step 6 *If  $H'/(H' \cap G'^0) \cong S_3$ , apply (O3) and restart the algorithm with the new pair.*

*Proof.* Step 1 is covered by Proposition 5.1. Moreover, each of the new pairs  $(H'', G'')$  produced in this step satisfies  $G''^0$  simple and  $C_{G''}(G''^0) = 1$ , so that the algorithm moves on to Step 2 after a possible application of Step 1.

From Step 2 on, we may assume that  $G'^0$  is simple and  $C_{G'}(G'^0) = 1$ . This allows us to identify  $H'/(H' \cap G'^0)$  with a subgroup of  $\text{Out}(G'^0)$ , and yields the constraints on its order. If  $p$  does not divide  $n$ , the quotient  $H'/(H' \cap G'^0)$  is linearly reductive. By Theorem 2.7(ii),  $H'$  is indeed  $G'$ -cr if and only if  $H' \cap G'^0$  is  $G'^0$ -cr.

From Step 3 on, we may assume in addition that  $p \in \{2, 3\}$  and that  $H$  is not contained in  $G^0$ . The conclusion of Step 3 is correct by Lemma 2.10(i).

Step 4 is an application of Corollary 4.5.

Since we have passed Step 3, we may assume that  $H' \cap G'^0$  is  $G'^0$ -cr. Under the condition of Step 5,  $H'/(H' \cap G'^0)$  is cyclic of prime order. The conclusion of Step 5 thus follows from Remark 2.12 and Theorem 3.3.

Finally, Step 6 is again covered by Proposition 5.1. Moreover, this step is only applicable for  $G'^0$  simple of type  $D_4$ . As we may assume  $H' \cap G'^0 \neq 1$  and  $Z(G'^0) = 1$ , the group  $M = H'C_{G'^0}(H' \cap G'^0)/(H' \cap G'^0)$  featuring in (O3) satisfies  $\dim M < \dim G'$ .

It remains to show that the algorithm terminates. Step 1 may restart finitely many instances of the algorithm. In each instance the algorithm terminates in Step 2 – Step 5 if Step 6 is not reached. If Step 6 is applicable, it replaces  $G'$ —which is simple of type  $D_4$ —with a group of smaller dimension. This implies that after Step 1 is applied again, Step 6 cannot be reached a second time, and the algorithm terminates.  $\square$

*Remark 5.4.* (i). It follows from the proof of Theorem 5.3 that Step 1, the only step that replaces a pair with several new pairs, need only be done at most twice along a path through the algorithm. Also, Step 6 only occurs at most once.

(ii). There are some situations where shortcuts may be applied to reduce to a connected group. First of all, if  $H^0$  is not reductive, then  $H$  cannot be  $G$ -cr. On the other hand, if  $H$  is cyclic, then we may apply Theorem 3.1 to deduce that  $H$  is  $G$ -cr if and only if  $C_{G^0}(H)$  is  $G^0$ -cr. Finally, if  $H/(H \cap G^0)$  is linearly reductive, we can apply Theorem 2.7(ii) to deduce that  $H$  is  $G$ -cr if and only if  $H \cap G^0$  is  $G^0$ -cr. However, the proposed algorithm gives a systematic approach that deals with all possible cases.

(iii). If  $p = 0$ , then a subgroup  $H$  is  $G$ -cr if and only if it is reductive ([14, Prop. 4.2]). Of course,  $H$  is reductive if and only if  $H^0$  is reductive, which in turn is equivalent to  $H^0$  being  $G^0$ -completely reducible.

## 6. EXAMPLES

We conclude with some examples of the algorithm outlined in Theorem 5.3.

**Example 6.1.** Let  $p = 3$ ,  $G = \text{Aut}(D_4)$ . Let  $\sigma$  be the triality graph automorphism as in Section 4. Let  $H = \langle \sigma \rangle K$ , where  $K = C_{D_4}(\sigma)$  is the fixed point subgroup of type  $G_2$ . We follow through the algorithm to deduce that  $H$  is  $G$ -cr:

Step 1 is not applicable, as  $G^0 = D_4$  is simple and  $C_G(G^0) = 1$ . In Step 2 we obtain  $n = 3 = p$  as the order of  $\langle \sigma \rangle \cong H/(H \cap G^0)$ . Now  $H \cap G^0 = K$  is  $G$ -cr (see Corollary 3.2), hence Steps 3 and 4 are not applicable. Step 5 applies and leads us to consider the group  $M = HC_{D_4}(K)/K \cong \langle \sigma \rangle C_{D_4}(K)$ . As  $K$  is adjoint, we obtain  $C_{M^0}(\sigma) = 1$  and thus this group is clearly  $M^0$ -cr. The algorithm stops with the conclusion that  $H$  is  $G$ -cr.

Here we have two commuting  $G$ -cr subgroups  $\langle \sigma \rangle$  and  $K$  of  $G$  and their product is also  $G$ -cr. This is not always the case: see [3, Ex. 5.1].

**Example 6.2.** Let  $\Gamma$  be a finite group acting transitively on a finite set  $I$ . Let  $i_0 \in I$ . Let  $\rho : \Gamma \rightarrow M$  be a homomorphism to a simple group  $M$  such that  $\rho(C_\Gamma(i_0))$  is not  $M$ -cr. We set

$$G = \Gamma \ltimes \prod_{i \in I} M,$$

where  $\Gamma$  acts on the product by permuting the indices. Clearly,  $G^0 = \prod_i M$ . Let  $d : M \rightarrow G^0$  be the diagonal embedding. As  $\Gamma$  commutes with the image of  $d$ , we may form the subgroup

$$H = \{\gamma d(\rho(\gamma)) \mid \gamma \in \Gamma\},$$

a finite subgroup of  $G$ . We claim that  $H$  is not  $G$ -cr, and use our algorithm to prove it.

To apply Step 1, we compute that for  $i \in I$ ,  $N_G(G_i) = C_\Gamma(i) \ltimes G^0$ . In particular,  $H_i = \{\gamma d(\rho(\gamma)) \mid \gamma \in C_\Gamma(i)\}$ . We obtain

$$\pi_i(H_i G^0) = H_i G^0 / \prod_{j \neq i} G_j \cong C_\Gamma(i) \times M =: G'_i,$$

and correspondingly

$$\pi_i(H_i) \cong \{\gamma \rho(\gamma) \mid \gamma \in C_\Gamma(i)\} =: H'_i.$$

Since the action of  $\Gamma$  on  $I$  is transitive, we may by Remark 5.2 replace  $(H, G)$  with  $(H', G')$ , where  $G' = G'_{i_0}$ ,  $H' = H'_{i_0}$ .

Now  $C_{G'}(G'^0) = C_\Gamma(i_0)$ , so Step 1 is again applicable and replaces  $(H', G')$  with the pair  $(H'', M)$ , where

$$H'' = \rho(C_\Gamma(i_0)).$$

By assumption,  $H''$  is not  $M$ -cr and hence  $H$  is not  $G$ -cr, by Step 2.

As a concrete realisation of this example, take  $\Gamma = \mathrm{PGL}_2(q)$  for  $q$  a sufficiently large power of  $p$ ,  $I = \mathrm{PGL}_2(q)/B(q)$  where  $B$  is a Borel subgroup of  $\mathrm{PGL}_2$ , and consider  $I$  as a transitive  $\Gamma$ -set by left translation. Let  $M = \mathrm{PGL}_2$ , and let  $\rho : \Gamma \hookrightarrow M$  the canonical embedding. If we take  $i_0 = B(q)$ , then  $C_\Gamma(i_0) = B(q)$  is not  $M$ -cr, for  $q$  large enough, as  $B$  is not  $M$ -cr. In this example, we have  $H \cap G^0 = 1$  and  $C_{G^0}(H) = 1$  (as  $C_M(B(q)) = 1$  for  $q$  sufficiently large). In particular, this example shows that  $H$  may fail to be  $G$ -cr even if  $C_{G^0}(H)$  and  $H \cap G^0$  both are  $G^0$ -cr.

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